Iterated Integrals

Suppose f is a function of two variables defined on rectangle $R = [a,b] \times [c,d]$.

For any fixed value x with $a \le x \le b$,

Define $A(x) = \int_c^d f(x,y)\,dy$ to be the result of integrating the one variable function $g(y)$ from c to d where, for all y with $c \le y \le d$, $g(y) = f(x,y)$

$$A(x) = \int_c^d g(y)\,dy = \int_c^d f(x,y)\,dy$$ Holding x as a fixed constant.

This is called the Partial Integral with respect to y.

The resulting function $A(x)$ can be integrated itself from a to b and this will equal the Double Integral:

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

EXAMPLE: For $f(x,y) = y^2 + 8xy^3 + x^3$,
and for Rectangle $[0,3] \times [1,2]$,
Let x be fixed at any value such that
$0 \le x \le 3$. Then the Partial Integral

$$A(x) = \int_1^2 f(x,y)\,dy = \int_1^2 (y^2 + 8xy^3 + x^3)\,dy$$

$$= \left[\tfrac{1}{3}y^3 + 2xy^4 + x^3 y\right]_1^2 \quad \begin{matrix} y=2 \\ y=1 \end{matrix}$$

$$= \left(\tfrac{1}{3}\cdot 8 + (2x)(16) + (x^3)(2)\right) - \left(\tfrac{1}{3} + 2x + x^3\right)$$

$$= x^3 + 30x + 7/3 = A(x).$$

Then

$$\iint_R (y^2 + 8xy^3 + x^3)\,dA = \int_0^3 A(x)\,dx$$

$$= \int_0^3 \left[\int_1^2 (y^2 + 8xy^3 + x^3)\,dy\right] dx$$

$$= \int_0^3 (x^3 + 30x + 7/3)\,dx = 162.25$$

EXAMPLE: This whole process can also be done by calculating $A(y) = \int_0^3 f(x,y)\,dx$, from fixing the value of y so that $1 \le y \le 2$, and then integrating $A(y)$ from 1 to 2 to get $\iint_R f(x,y)\,dA$.

Thus, $\iint_R (y^2+8xy^3+x^3)\,dA$

$$= \int_1^2 \left[\int_0^3 (y^2+8xy^3+x^3)\,dx \right] dy$$

$$= \int_1^2 \left(\left[y^2x + 4x^2y^3 + \tfrac{1}{4}x^4 \right]_{x=0}^{x=3} \right) dy$$

$$= \int_1^2 \left(\left(3y^2+36y^3+\tfrac{81}{4}\right) - (0) \right) dy$$

$$= \int_1^2 (3y^2+36y^3+81/4)\,dy = \left[y^3 + 9y^4 + \tfrac{81}{4}y \right]_1^2$$

$$= \left(8+144+\tfrac{81}{2}\right) - \left(1+9+\tfrac{81}{4}\right) = 162\tfrac{1}{4} = 162.25.$$

Ex: Determine the Volume V under the surface of the graph and above the rectangle $[0,1]\times[0,1]$ for the function $f(x,y) = 16 - x^2 - y^2$.

$$V = \iint_{[0,1]\times[0,1]} (16-x^2-y^2)\,dA = \int_0^1\int_0^1 (16-x^2-y^2)\,dy\,dx$$

$$= \int_0^1 \left[\int_0^1 (16-x^2-y^2)\,dy\right] dx$$

$$= \int_0^1 \left[\left[16y - x^2y - \frac{1}{3}y^3\right]_{y=0}^{y=1} \right] dx$$

$$= \int_0^1 \left[\left(16 - x^2 - \frac{1}{3}\right) - (0)\right] dx = \int_0^1 \left(16 - x^2 - \frac{1}{3}\right) dx$$

$$= \left[16x - \frac{1}{3}x^3 - \frac{1}{3}x\right]_0^1 = \left(16 - \frac{1}{3} - \frac{1}{3}\right) - (0)$$

$$= 16 - 2/3 = 15\tfrac{1}{3} = 46/3$$

The Volume $V = 15\tfrac{1}{3}$ cubic units

<u>Ex</u>: Evaluate

$$I = \int_2^4 \int_{-1}^1 (x^2+y^2)\,dy\,dx$$

$$= \int_2^4 \left[\int_{-1}^1 (x^2+y^2)\,dy \right] dx$$

$$= \int_2^4 \left(\left[x^2 y + \frac{1}{3} y^3 \right]_{-1}^1 \right) dx$$

$$= \int_2^4 \left(\left(x^2 + \frac{1}{3}\right) - \left(-x^2 - \frac{1}{3}\right) \right) dx$$

$$= \int_2^4 \left(2x^2 + 2/3\right) dx = \left[\frac{2}{3} x^3 + \frac{2}{3} x \right]_2^4$$

$$= \left(\frac{128}{3} + \frac{8}{3} \right) - \left(\frac{16}{3} + \frac{4}{3} \right) = \frac{116}{3} = 38\,2/3$$